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ALMOST SURE CONVERGENCE OF THE BARTLETT ESTIMATOR

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Dedicated to Endre Csáki and Pál Révész
on the occasion of their 70^{th} birthdays

Abstract

We study the almost sure convergence of the Bartlett estimator for the asymptotic variance of the sample mean of a stationary weekly dependent process. We also study the a. s. behavior of this estimator in the case of long-range dependent observations. In the weakly dependent case, we establish conditions under which the estimator is strongly consistent. We also show that, after appropriate normalization, the estimator converges a.s. in the long-range dependent case as well. In both cases, our conditions involve fourth order cumulants and assumptions on the rate of growth of the truncation parameter appearing in the definition of the Bartlett estimator.

1. Introduction

If $\{Y_i\}$ is a weakly stationary sequence, then under weak conditions which quantify short-range (weak) dependence

$$\frac{1}{n}\mathrm{Var}\left(\sum_{1\le i\le n} Y_i\right) \to \sigma^2 := \sum_{j=-\infty}^{\infty} \gamma_j,$$

as $n \to \infty$, where $\gamma_j = \mathrm{Cov}(Y_0, Y_j)$. Inference for time series modeled by weakly dependent processes requires estimation of the asymptotic variance σ^2. One of the

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most popular estimators is the Bartlett estimator defined as

$$s_n^2 = \hat{\gamma}_0 + 2 \sum_{1 \le j \le q(n)} \omega_j(q(n)) \hat{\gamma}_j, \tag{1.1}$$

where

$$\hat{\gamma}_j = \frac{1}{n} \sum_{1 \le i \le n-j} \big(Y_i - \bar{Y}_n\big)\big(Y_{i+j} - \bar{Y}_n\big) \tag{1.2}$$

are the sample autocovariances and $\omega_j(q)$ are the Bartlett weights defined by

$$\omega_j(q) = 1 - \frac{j}{q+1}. \tag{1.3}$$

The estimator s_n^2 has also been used for long-range dependent observations Y_i, see Lo (1991). In the weakly dependent case, $\sigma^2 = 2\pi f(0)$, where f is the spectral density of $\{Y_i\}$, so $s_n^2/(2\pi)$ is also an estimator for $f(0)$. An extension of the estimator (1.1) to arbitrary frequencies has been studied extensively in the spectral domain, so most results focused on L^2 convergence. Giraitis *et al.* (2003) proved that under regularity conditions quantifying weak dependence, $s_n^2 \xrightarrow{P} \sigma^2$. They also considered the in probability behaviour of s_n^2 when $\{Y_i\}$ exhibit long range dependence. In the present paper we establish the almost sure consistency of the variance estimator s_n^2 under the conditions used by Giraitis *et al.* (2003) in the case of weak dependence as well as in the long memory case. Theorem 1.1 below plays a crucial role in Berkes *et al.* (2003) who developed a procedure for distinguishing between a sequence of long-range dependent observations and a sequence of weakly dependent observations with a change point.

To lighten the notation, we assume in the following that $EY_i = 0$.

Recall the definition of the fourth order cumulant $\kappa(h,r,s)$ given by

$$\kappa(h,r,s) = E[Y_k Y_{k+h} Y_{k+r} Y_{k+s}] - (\gamma_h \gamma_{r-s} + \gamma_r \gamma_{h-s} + \gamma_s \gamma_{h-r}). \tag{1.4}$$

We will also work with the quantity

$$\begin{aligned} \nu(h,r,s) &= \mathrm{Cov}(Y_k Y_{k+h}, Y_{k+r} Y_{k+s}) \\ &= E[Y_k Y_{k+h} Y_{k+r} Y_{k+s}] - \gamma_h \gamma_{r-s} \\ &= \kappa(h,r,s) + \gamma_r \gamma_{h-s} + \gamma_s \gamma_{h-r}. \end{aligned} \tag{1.5}$$

THEOREM 1.1. *Suppose* $\{Y_k\}$ *is a fourth order stationary sequence with* $EY_i = 0$ *and* $\gamma_j = \mathrm{Cov}(Y_0, Y_j)$.

Suppose the sequence $q(n)$ *is nondecreasing and*

$$\sup_{k \ge 0} \frac{q\left(2^{k+1}\right)}{q\left(2^k\right)} < \infty. \tag{1.6}$$

(i) Suppose, in addition, that

$$\sum_j |\gamma_j| < \infty, \tag{1.7}$$

$$\sup_h \sum_{r,s} |\kappa(h,r,s)| < \infty, \tag{1.8}$$

and

$$q(n) \to \infty \quad \text{and} \quad q(n)(\log n)^4 = O(n). \tag{1.9}$$

Then,

$$s_n^2 \to \sigma^2 := \sum_{j=-\infty}^{\infty} \gamma_j \quad \textit{a.s.} \tag{1.10}$$

(ii) Assume

$$\frac{1}{2} < H < 1 \tag{1.11}$$

and

$$\gamma_k \sim c_0 k^{2H-2} \tag{1.12}$$

for some $c_0 > 0$. Assume also that

$$q(n) \to \infty \quad \textit{and} \quad q(n) = O\left(n(\log n)^{-7/(4-4H)}\right) \tag{1.13}$$

and

$$\sup_{|h| \le q(n)} \sum_{-n \le r,s \le n} |\kappa(h,r,s)| = O\left(n^{2H-1}\right). \tag{1.14}$$

Then

$$q(n)^{1-2H} s_n^2 \to c_H^2 = \frac{c_0}{H(2H-1)} \quad \textit{a.s.} \tag{1.15}$$

Remark 1.1. The proof of Theorem 1.1 remains valid if the Bartlett weights (1.3) are replaced by arbitrary weights which in addition to (1.6) satisfy $\omega_j(q) = 0$ for $|j| > q$, $0 \le \omega_j(q) \le 1$ and another condition which is different for parts (i) and (ii). For part (i), it must be required (cf. (2.5)) that

$$\lim_{q \to \infty} \omega_j(q) = 1 \quad \text{for each } j. \tag{1.16}$$

For part (ii), it must be required that

$$\lim_{q \to \infty} q^{1-2H} \sum_{|j| \le q} \omega_j(q) \tilde{\gamma}_j = \frac{c_0}{(2H-1)H}. \tag{1.17}$$

2. Proofs

Proof of part (i). Let

$$\tilde{\gamma}_j = \frac{1}{n} \sum_{i=1}^{n-|j|} Y_i Y_{i+|j|}, \quad |j| < n.$$

Define also

$$S_{k,l} = \sum_{i=k}^{l} Y_i.$$

Then

$$\hat{\gamma}_j - \tilde{\gamma}_j = \left(1 - \frac{|j|}{n}\right) \bar{Y}_n^2 - \frac{1}{n} \bar{Y}_n (S_{1,n-|j|} + S_{|j|+1,n}) =: \delta_j.$$

As in the proof of Theorem 3.1 of Giraitis *et al.* (2003), decompose s_n^2 as

$$s_n^2 = v_{n,1} + v_{n,2},$$

where

$$v_{n,1} = \sum_{|j| \le q(n)} \left(1 - \frac{|j|}{q(n)+1}\right) \tilde{\gamma}_j,$$

$$v_{n,2} = \sum_{|j| \le q(n)} \left(1 - \frac{|j|}{q(n)+1}\right) \delta_j.$$

It suffices to show that

$$v_{n,1} \to \sigma^2 \quad a.s. \tag{2.1}$$

and

$$v_{n,2} \to 0 \quad a.s. \tag{2.2}$$

We first verify the easier relation (2.2). By (1.7) and Theorem 3.7.2 of Stout (1974)

$$|S_{1,n}| = o\left(n^{1/2} \log^2 n\right) \quad a.s. \tag{2.3}$$

and consequently

$$\max_{1 \le i \le j \le n} |S_{i,j}| = o\left(n^{1/2} \log^2 n\right) \quad a.s.$$

Hence, by (1.9),

$$\begin{aligned} \sum_{|j| \le q(n)} \left(1 - \frac{|j|}{q(n)+1}\right) \left(1 - \frac{|j|}{n}\right) \bar{Y}_n^2 &\le (2q(n)+1) \bar{Y}_n^2 \\ &= o\left((q(n)/n) \log^4 n\right) \\ &= o(1) \quad a.s. \end{aligned}$$

and

$$\begin{aligned}\sum_{|j|\le q(n)} \left(1-\frac{|j|}{q(n)+1}\right)\frac{1}{n}\left|\bar{Y}_n\right|\left|S_{1,n-|j|}+S_{|j|+1,n}\right| & \\ &\le \frac{2q(n)+1}{n}\frac{o\left(\log^2 n\right)}{n^{1/2}}\left(n^{1/2}\log^2 n\right)\\ &= o\left(\frac{q(n)\log^4 n}{n}\right) = o(1) \quad a.s.\end{aligned}$$

This proves (2.2).

We now turn to the verification of (2.1). Set

$$z_n = n(v_{n,1}-Ev_{n,1}) = \sum_{|j|\le q(n)}\left(1-\frac{|j|}{q(n)+1}\right)\sum_{i=1}^{n-|j|}\left(Y_iY_{i+|j|}-\gamma_{|j|}\right). \tag{2.4}$$

Since

$$Ev_{n,1} = \sum_{|j|\le q(n)}\left(1-\frac{|j|}{q(n)+1}\right)\left(1-\frac{|j|}{n}\right)\gamma_j \to \sum_{j=-\infty}^{\infty}\gamma_j = \sigma^2, \tag{2.5}$$

it suffices to verify that $z_n = o(n)$ *a.s.* In the study of z_n, we replace $q(n)$ with $q^*(n)$, where $q^*(n)$ is constant on large intervals. This replacement will be done in two steps. First we replace $q(n)$ in the limit of summation and then in the sum itself. We thus define

$$q^*(n) = q(2^k), \quad \text{if } 2^k < n \le 2^{k+1}$$

and introduce

$$\tilde{z}_n = \sum_{|j|\le q^*(n)}\left(1-\frac{|j|}{q(n)+1}\right)\sum_{i=1}^{n-|j|}\left(Y_iY_{i+|j|}-\gamma_{|j|}\right), \tag{2.6}$$

$$\hat{z}_n = \sum_{|j|\le q^*(n)}\left(1-\frac{|j|}{q^*(n)+1}\right)\sum_{i=1}^{n-|j|}\left(Y_iY_{i+|j|}-\gamma_{|j|}\right). \tag{2.7}$$

We will show that

$$z_n - \tilde{z}_n = o(n) \quad a.s., \tag{2.8}$$

$$\tilde{z}_n - \hat{z}_n = o(n) \quad a.s. \tag{2.9}$$

and

$$\hat{z}_n = o(n) \quad a.s. \tag{2.10}$$

We will use the relation

$$\sup_h \sum_{r,s} |\nu(h,r,s)| < \infty \tag{2.11}$$

which follows immediately from (1.8) and (1.7).

Clearly,

$$\begin{aligned} z_n - \tilde{z}_n &= \sum_{q^*(n)<|j|\le q(n)} \left(1 - \frac{|j|}{q(n)+1}\right) \sum_{i=1}^{n-|j|} \left(Y_i Y_{i+|j|} - \gamma_{|j|}\right) \\ &= \Delta_{n,1} + \Delta_{n,2}, \end{aligned} \tag{2.12}$$

where

$$\Delta_{n,1} = \sum_{q^*(n)<|j|\le q(n)} \left(1 - \frac{|j|}{q^*(n)+1}\right) \sum_{i=1}^{n-|j|} \left(Y_i Y_{i+|j|} - \gamma_{|j|}\right)$$

and

$$\Delta_{n,2} = \sum_{q^*(n)<|j|\le q(n)} |j| \left(\frac{1}{q^*(n)+1} - \frac{1}{q(n)+1}\right) \sum_{i=1}^{n-|j|} \left(Y_i Y_{i+|j|} - \gamma_{|j|}\right).$$

We first show that

$$\Delta_{n,1} = o(n) \quad a.s. \tag{2.13}$$

For any $2^k < m < n \le 2^{k+1}$, we write

$$\begin{aligned} \Delta_{n,1} - \Delta_{m,1} &= \sum_{q^*(n)<|j|\le q(m)} \left(1 - \frac{|j|}{q^*(n)+1}\right) \sum_{i=m-|j|+1}^{n-|j|} \left(Y_i Y_{i+|j|} - \gamma_{|j|}\right) \\ &\quad + \sum_{q(m)<|j|\le q(n)} \left(1 - \frac{|j|}{q^*(n)+1}\right) \sum_{i=1}^{n-|j|} \left(Y_i Y_{i+|j|} - \gamma_{|j|}\right) \\ &=: a_1(n,m) + a_2(n,m). \end{aligned} \tag{2.14}$$

Using (2.11), we have

$$\begin{aligned} &E[a_1(n,m)]^2 \\ &\le \sum_{q^*(n)<|j|\le q(m)} \ \sum_{q^*(n)<|j'|\le q(m)} \ \sum_{m-|j|<i\le n-|j|} \ \sum_{m-|j'|<i'\le n-|j'|} |\mathrm{Cov}(Y_i Y_{i+|j|},\ Y_{i'} Y_{i'+|j'|})| \\ &= \sum_{q^*(n)<|j|\le q(m)} \ \sum_{q^*(n)<|j'|\le q(m)} \ \sum_{m-|j|<i\le n-|j|} \ \sum_{m-|j'|<i'\le n-|j'|} |\nu(|j|,\ i'-i,\ i'-i+|j'|)| \\ &= \sum_{q^*(n)<|j|\le q(m)} \ \sum_{q^*(n)<|j'|\le q(m)} \ \sum_{m<i<n} \ \sum_{m<i'<n} |\nu(|j|,\ i'-i-|j'|+|j|,\ i'-i+|j|)| \\ &\le (n-m) \sum_{|j|,|j'|\le q(m)} \ \sum_{|l|\le 4n} |\nu(|j|,\ l,\ l+|j'|)| \\ &\le C(n-m)q(m). \end{aligned} \tag{2.15}$$

Applying (2.11) again, we obtain

$$\begin{aligned}
&E[a_2(n,m)]^2 \\
&\leq \sum_{q(m)<|j|\leq q(n)} \sum_{q(m)<|j'|\leq q(n)} \sum_{i=1}^{n-|j|} \sum_{i'=1}^{n-|j'|} |\mathrm{Cov}(Y_i Y_{i+|j|},\ Y_{i'} Y_{i'+|j'|})| \\
&= \sum_{q(m)<|j|\leq q(n)} \sum_{q(m)<|j'|\leq q(n)} \sum_{i=1}^{n-|j|} \sum_{i'=1}^{n-|j'|} |\nu(|j|,\ i'-i,\ i'-i+|j'|)| \\
&\leq n \sum_{q(m)<|j|\leq q(n)} \sum_{q(m)<|j'|\leq q(n)} \sum_{|l|\leq 4n} |\nu(|j|,\ l,\ l+|j'|)| \\
&\leq C(q(n)-q(m))n. \qquad (2.16)
\end{aligned}$$

Combining (2.14)–(2.16), we conclude that

$$\begin{aligned}
E[\Delta_{n,1}-\Delta_{m,1}]^2 &\leq C[(n-m)q(n)+(q(n)-q(m))n] \\
&= C\sum_{i=m+1}^{n} [q(n)+(q(i)-q(i-1))n] \\
&\leq C\sum_{i=m+1}^{n} \left[q\left(2^{k+1}\right)+(q(i)-q(i-1))2^{k+1}\right]. \qquad (2.17)
\end{aligned}$$

Using Problem 5 of Billingsley (1968), p. 102, (cf. Móricz *et al.* (1982)), we get

$$\begin{aligned}
&E\max_{2^k<m\leq 2^{k+1}} \left[\Delta_{2^{k+1},1}-\Delta_{m,1}\right]^2 \\
&\leq C\left[\left(2^{k+1}-2^k\right)q\left(2^{k+1}\right)+\left(q\left(2^{k+1}\right)-q\left(2^k\right)\right)2^{k+1}\right]\left(\log 2^{k+1}\right)^2,
\end{aligned}$$

and therefore the Chebishev inequality and (1.6) yield

$$\begin{aligned}
&P\left\{\max_{2^k<m\leq 2^{k+1}} |\Delta_{2^{k+1},1}-\Delta_{m,1}| \geq k^{7/4} q^{1/2}(2^k) 2^{k/2}\right\} \\
&\leq C\frac{\left[\left(2^{k+1}-2^k\right)q\left(2^{k+1}\right)+\left(q\left(2^{k+1}\right)-q\left(2^k\right)\right)2^{k+1}\right]k^2}{k^{7/2}q\left(2^k\right)2^k} \leq \frac{C}{k^{3/2}}.
\end{aligned}$$

The Borel–Cantelli lemma and (1.9) give

$$\max_{2^k<m\leq 2^{k+1}} |\Delta_{2^{k+1},1}-\Delta_{m,1}| \overset{a.s.}{=} o\left(2^k\right). \qquad (2.18)$$

Similarly to (2.15), we have, using (2.11)

$$\begin{aligned}
E\Delta_{n,1}^2 &\le \sum_{q^*(n)<|j|,|j'|\le q(n)} \ \sum_{1\le i\le n-|j|} \ \sum_{1\le i'\le n-|j'|} \left|\operatorname{Cov}\left(Y_iY_{i+|j|},\ Y_{i'}Y_{i'+|j'|}\right)\right| \\
&= \sum_{q^*(n)<|j|,|j'|\le q(n)} \ \sum_{1\le i\le n-|j|} \ \sum_{1\le i'\le n-|j'|} |\nu(|j|,\ i'-i,\ i'-i+|j'|)| \\
&= \sum_{q^*(n)<|j|,|j'|\le q(n)} \ \sum_{1\le i,i'\le n} |\nu(|j|,\ i'-i-|j'|+|j|,\ i'-i+|j|)| \\
&\le n \sum_{q^*(n)<|j|,|j'|\le q(n)} \ \sum_{|l|\le 4n} |\nu(|j|,\ l,\ l+|j'|)| \\
&\le Cnq(n). \qquad (2.19)
\end{aligned}$$

Hence the Chebishev inequality and (1.6) give

$$P\left\{\Delta_{2^{k+1},1} \ge k^{7/4}q^{1/2}(2^k)2^{k/2}\right\} \le C\frac{2^{k+1}q(2^{k+1})}{k^{7/2}q(2^k)2^k} \le \frac{C}{k^{7/2}},$$

so the Borel–Cantelli lemma and (1.9) yield

$$\Delta_{2^{k+1},1} \overset{a.s.}{=} o(2^{k+1}). \qquad (2.20)$$

Now (2.13) follows from (2.18) and (2.20).

Next we show

$$\Delta_{n,2} = o(n) \quad a.s. \qquad (2.21)$$

Observe that

$$\begin{aligned}
|\Delta_{n,2}| &= \left|\frac{q(n)-q^*(n)}{q(n)+1}\right| \left| \sum_{q^*(n)<|j|\le q(n)} \frac{|j|}{q^*(n)+1} \sum_{i=1}^{n-|j|} \left(Y_iY_{i+|j|}-\gamma_{|j|}\right) \right| \\
&\le \left| \sum_{q^*(n)<|j|\le q(n)} \frac{|j|}{q^*(n)+1} \sum_{i=1}^{n-|j|} \left(Y_iY_{i+|j|}-\gamma_{|j|}\right) \right|.
\end{aligned}$$

For any $2^k < m < n \le 2^{k+1}$, we write

$$\begin{aligned}
\Delta_{n,2}-\Delta_{m,2} = &\sum_{q^*(n)<|j|\le q(n)} \frac{|j|}{q^*(n)+1} \sum_{i=1}^{n-|j|} \left(Y_iY_{i+|j|}-\gamma_{|j|}\right) \\
&- \sum_{q^*(n)<|j|\le q(m)} \frac{|j|}{q^*(n)+1} \sum_{i=1}^{m-|j|} \left(Y_iY_{i+|j|}-\gamma_{|j|}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{q^*(n)<|j|\le q(m)} \frac{|j|}{q^*(n)+1} \sum_{i=m-|j|+1}^{n-|j|} \left(Y_iY_{i+|j|}-\gamma_{|j|}\right)\\
&\quad + \sum_{q(m)<|j|\le q(n)} \frac{|j|}{q^*(n)+1} \sum_{i=1}^{n-|j|} \left(Y_iY_{i+|j|}-\gamma_{|j|}\right)\\
&= a_3(n,m)+a_4(n,m).
\end{aligned}
$$

Following the proofs of (2.15) and (2.16), we obtain

$$E[a_3(n,m)]^2 \le C(n-m)q(m), \tag{2.22}$$

$$E[a_4(n,m)]^2 \le C(q(n)-q(m))n \tag{2.23}$$

and

$$E\Delta_{n,2}^2 \le Cnq(n). \tag{2.24}$$

Just as relations (2.15), (2.16) and (2.19) implied (2.13), relations (2.22), (2.23) and (2.24) imply (2.21).

Relation (2.8) follows from (2.13) and (2.21).

Observing that

$$\tilde{z}_n - \hat{z}_n = \frac{q(n)-q^*(n)}{q(n)+1} \sum_{|j|\le q^*(n)} \frac{|j|}{q^*(n)+1} \sum_{i=1}^{n-|j|} \left(Y_iY_{i+|j|}-\gamma_{|j|}\right),$$

and following the proof of (2.21), one can easily verify that (2.9) holds.

Also, similarly to (2.15) and (2.24), for any $2^k < m, n \le 2^{k+1}$, we have

$$E[\hat{z}_n - \hat{z}_m]^2 \le C(n-m)q^*(n)$$

and

$$E\hat{z}_n^2 \le Cnq^*(n).$$

Hence, relation (2.10) follows from Problem 5 of Billingsley (1968), the Chebyshev inequality, (1.6) and the Borel–Cantelli lemma.

PROOF OF PART (ii). The idea of the proof is similar to that used in part (i) but different bounds are needed. We use the same notation as in the proof of part (i).

As verified on p. 291 of Giraitis *et al.* (2003)

$$q(n)^{1-2H}Ev_{n,1} = q(n)^{1-2H} \sum_{|j|\le q(n)} \omega_j(q(n))\tilde{\gamma}_j \to \frac{c_0}{H(2H-1)} = c_H^2. \tag{2.25}$$

We first show that $q(n)^{1-2H}v_{n,2} \to 0$ *a.s.* Observe that by (1.12) it follows easily

$$ES_{k,l}^2 \le C(l-k+1)^{2H}$$

and hence

$$E \max_{1\le l\le n} |S_{0,l}|^2 \le Cn^{2H} \tag{2.26}$$

by a maximal inequality of Billingsley (1968), p. 94. We will now verify that (2.26) implies

$$\max_{1\le l\le n} |S_{0,l}| = o\left(n^H \log n\right) \quad a.s. \tag{2.27}$$

Fix $1/2 < p < 1$ and note that for any $\epsilon > 0$

$$\begin{aligned}\sum_{1\le k<\infty} P\left(\max_{1\le l\le 2^k} |S_{0,l}| > \epsilon 2^{kH} k^p\right) &\le \frac{1}{\epsilon^2} \sum_{1\le k<\infty} \left[2^{kH}k^p\right]^{-2} E \max_{1\le l\le 2^k} |S_{0,l}|^2 \\ &\le \frac{1}{\epsilon^2} \sum_{1\le k<\infty} \left[2^{kH}k^p\right]^{-2} C 2^{2kH} \\ &= C\frac{1}{\epsilon^2} \sum_{1\le k<\infty} k^{-2p} < \infty.\end{aligned}$$

Hence, by the Borel–Cantelli lemma

$$\max_{1\le l\le 2^k} |S_{0,l}| = o\left(2^{kH}k^p\right) \quad a.s.$$

as $k \to \infty$. Now, for any n, choosing k such that $2^{k-1} < n \le 2^k$, we obtain

$$\max_{1\le l\le n} |S_{0,l}| \le \max_{1\le l\le 2^k} |S_{0,l}| = o\left(2^{kH}k^p\right) = o\left(n^H \log n\right) \quad a.s.,$$

establishing (2.27).

Using the definition of $v_{n,2}$ and (2.27), it is easy to see that

$$\begin{aligned}q(n)^{1-2H} v_{n,2} &= o(1) q(n)^{2-2H} (\log^2 n) n^{2H-2} \\ &= o(1) \left[\frac{q(n)}{n} (\log n)^{1/(1-H)}\right]^{2-2H} \quad a.s.\end{aligned}$$

Therefore, assumption (1.13) implies that $q(n)^{1-2H} v_{n,2} \to 0$ *a.s.*

We now show that $q(n)^{1-2H}(v_{n,1} - Ev_{n,1}) \to 0$ *a.s.* This will be accomplished by showing that

$$z_n - \tilde{z}_n = o\left(nq(n)^{2H-1}\right) \quad a.s., \tag{2.28}$$

$$\tilde{z}_n - \hat{z}_n = o\left(nq(n)^{2H-1}\right) \quad a.s. \tag{2.29}$$

and

$$\hat{z}_n = o\left(nq(n)^{2H-1}\right) \quad a.s., \tag{2.30}$$

with z_n, $\tilde{z}_n$ and $\hat{z}_n$ defined, respectively, by (2.4), (2.6) and (2.7).

We obtain some inequalities for the second moments of $z_n - \tilde{z}_n$, $\tilde{z}_n - \hat{z}_n$ and $\hat{z}_n$ and their increments on the intervals $\left(2^k, 2^{k+1}\right]$. The inequality in Problem 5 of Billingsley (1968), p. 102 (cf. also Corollary 3.1 in Móricz *et al.* (1982)) with the Borel–Cantelli lemma then yields (2.28)–(2.30).

First we use the decompositions in (2.12) and (2.14). Similarly to (2.15) , for $2^k < m < n \le 2^{k+1}$, we have, using the first relation of Lemma 2.1,

$$\begin{aligned} E[a_1(n,m)]^2 &\le (n-m) \sum_{|j|\le q(m)} \; \sum_{|j'|\le q(m)} \; \sum_{|l|\le 4n} |\nu(|j|,\ l,\ l+|j'|)| \\ &\le C(n-m)q(n)^{2H} n^{2H-1}. \end{aligned} \tag{2.31}$$

As in (2.16), we have, using the second relation of Lemma 2.1,

$$\begin{aligned} E[a_2(n,m)]^2 &\le n \sum_{q(m)<|j|\le q(n)} \; \sum_{q(m)<|j'|\le q(n)} \; \sum_{|l|\le 4n} |\nu(|j|,\ l,\ l+|j'|)| \\ &\le C\left((q(n)-q(m))n^{2H} q(n)^{2H-1}\right). \end{aligned} \tag{2.32}$$

Hence, using also (1.13) we get

$$\begin{aligned} &P\left\{ \max_{2^k<m\le 2^{k+1}} \left|\Delta_{2^{k+1},1} - \Delta_{m,1}\right| \ge \epsilon 2^{k+1} \left[q\left(2^{k+1}\right)\right]^{2H-1} \right\} \\ &\le \frac{Ck^2}{\epsilon^2} \frac{\left(2^{k+1}-2^k\right)\left[q\left(2^{k+1}\right)\right]^{2H} 2^{(2H-1)k} + \left(q\left(2^{k+1}\right) - q\left(2^k\right)\right)\left[q\left(2^{k+1}\right)\right]^{2H-1} 2^{2kH}}{2^{2k}\left[q\left(2^{k+1}\right)\right]^{4H-2}} \\ &\le \frac{Ck^2}{\epsilon^2} \left(\frac{q\left(2^{k+1}\right)}{2^{k+1}}\right)^{2-2H} \le \frac{Ck^{-3/2}}{\epsilon^2}, \end{aligned} \tag{2.33}$$

so by the Borel–Cantelli lemma we have

$$\max_{2^k<m\le 2^{k+1}} \left|\Delta_{2^{k+1},1} - \Delta_{m,1}\right| = o\left(2^{k+1}\left[q\left(2^{k+1}\right)\right]^{2H-1}\right) \quad a.s. \tag{2.34}$$

Similarly,

$$E\Delta_{n,1}^2 \le C(nq(n))^{2H},$$

so by (1.6), (1.13) and the Borel–Cantelli lemma we have

$$\Delta_{2^{k+1},1} \overset{a.s.}{=} o\left(2^{k+1}\left(q\left(2^{k+1}\right)\right)^{2H-1}\right).$$

Hence we conclude that

$$\Delta_{n,1} = o\left(nq(n)^{2H-1}\right) \quad a.s. \tag{2.35}$$

Similarly, along the lines of the proof of (2.21) and (2.35), we get

$$\Delta_{n,2} = o\left(nq(n)^{2H-1}\right) \quad a.s., \tag{2.36}$$

completing the verification of (2.28).

The proofs of (2.29) and (2.30) are similar to that of (2.28) and are therefore omitted.

We conclude this section with Lemma 2.1 which was used in the proof of Theorem 1.1. The proof of Lemma 2.1 uses Lemma 2.2 which follows the proof of Lemma 2.1.

LEMMA 2.1. *Suppose* $\{X_k\}$ *is a fourth order stationary sequence. If (1.12) is satisfied with* $1/2 < H < 1$ *and assumptions (1.13) and (1.14) hold, then*

$$\sum_{|h|\le q(n)} \sum_{|s|\le q(n)} \sum_{|r|\le 4n} |\nu(h,r,r+|s|)| = O\left(q(n)^{2H} n^{2H-1}\right) \tag{2.37}$$

and

$$\sum_{q(m)<|h|\le q(n)} \sum_{q(m)<|s|\le q(n)} \sum_{|r|\le 4n} |\nu(|h|,r,r+|s|)| = O\left((q(n)-q(m)) n^{2H-1} q(n)^{2H-1}\right). \tag{2.38}$$

PROOF. We note that by (1.6), relation (1.14) remains valid if the summation domain $-n \le r,s \le n$ is changed to $-4n \le r,s \le 4n$. Thus

$$\sum_{|h|\le q(n)} \sum_{|s|\le q(n)} \sum_{|r|\le 4n} |\kappa(h,r,r+|s|)| = O\left(q(n) n^{2H-1}\right),$$

and by Lemma 2.2

$$\sum_{|h|\le q(n)} \sum_{|s|\le q(n)} \sum_{|r|\le 4n} \left[\left|\gamma_r \gamma_{|h|-(r+|s|)}\right| + \left|\gamma_{r+|s|}\gamma_{|h|-r}\right| \right]$$

$$\le q(n) \left[\sup_h \sum_{|s|\le q(n)} \left|\gamma_{|h|-(r+|s|)}\right| \sum_{|r|\le 4n} |\gamma_r| + \sup_h \sum_{|r|\le 4n} \left|\gamma_{|h|-r}\right| \sum_{|s|\le q(n)} \left|\gamma_{r+|s|}\right| \right]$$

$$= O\left(q(n)^{2H} n^{2H-1}\right).$$

Since $H > 1/2$, we have $q(n) = o\left(q(n)^{2H}\right)$, and so (2.37) follows from the identity (1.5).

Using (1.14) again, we have

$$\sum_{q(m)<|h|\le q(n)} \sum_{q(m)<|s|\le q(n)} \sum_{|r|\le 4n} |\kappa(|h|,r,r+|s|)| = O\left((q(n)-q(m))n^{2H-1}\right)$$

and by Lemma 2.2,

$$\begin{aligned}
&\sum_{q(m)<|h|\le q(n)} \sum_{q(m)<|s|\le q(n)} \sum_{|r|\le 4n} \left|\gamma_r\gamma_{|h|-(r+|s|)}\right| \\
&= 2 \sum_{q(m)<|h|\le q(n)} \sum_{q(m)<s\le q(n)} \sum_{|r|\le 4n} \left|\gamma_r\gamma_{|h|-s-r}\right| \\
&\le 2 \sum_{q(m)<|h|\le q(n)} \sum_{|i|\le 2q(n)} \sum_{|r|\le 4n} |\gamma_r\gamma_{i-r}| \\
&\le C(q(n)-q(m))n^{2H-1}q(n)^{2H-1}.
\end{aligned}$$

Similarly, by (1.12) and Lemma 2.2,

$$\begin{aligned}
&\sum_{q(m)<|h|\le q(n)} \sum_{q(m)<|s|\le q(n)} \sum_{|r|\le 4n} \left|\gamma_{r+|s|}\gamma_{|h|-r}\right| \\
&= \sum_{q(m)<|h|\le q(n)} \sum_{q(m)<|s|\le q(n)} \left(\sum_{0\le r\le 4n} + \sum_{-4n\le r\le -1}\right) \left|\gamma_{r+|s|}\gamma_{|h|-r}\right| \\
&\le C(q(n)-q(m)) \left[\left(1+\sum_{1\le r\le 4n} r^{2H-2}\right) \sum_{q(m)<|h|\le q(n)} \left|\gamma_{|h|-r}\right| \right. \\
&\qquad \left. + \sum_{-4n\le r\le -1} |r|^{2H-2} \sum_{q(m)<|s|\le q(n)} \left|\gamma_{r+|s|}\right|\right] \\
&\le C(q(n)-q(m))n^{2H-1}q(n)^{2H-1}.
\end{aligned}$$

Lemma 2.2. *Suppose* $\{X_k\}$ *is a fourth order stationary sequence. If (1.12) is satisfied with* $1/2 < H < 1$, *then*

$$\sum_{i=1}^{k} |\gamma_{i+v}| \le Ck^{2H-1}$$

uniformly in $v = 0, \pm 1, \pm 2, \ldots$.

PROOF. Let $v \geq 1$, then

$$\sum_{i=1}^{k} |\gamma_{i+v}| \leq C \sum_{i=1}^{k} |i+v|^{2H-2}$$

$$\leq C \sum_{i=1}^{k} |i|^{2H-2} \leq Ck^{2H-1}.$$

Let $v \leq 0$. If $v \leq -k-1$ then we have

$$\sum_{i=1}^{k} |\gamma_{i+v}| \leq C \sum_{i=1}^{k} |i+v|^{2H-2} \leq C \sum_{i=1}^{k} |i-k-1|^{2H-2}$$

$$\leq C \sum_{i=1}^{k} |i|^{2H-2} \leq Ck^{2H-1}.$$

If $-k-1 \leq v \leq 0$, we have

$$\sum_{i=1}^{k} |\gamma_{i+v}| \leq 2C \sum_{i=1}^{2k} |i|^{2H-2} \leq Ck^{2H-1}.$$

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