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Convergence of quadratic forms with nonvanishing diagonal

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Abstract

Motivated by applications to time series analysis, we establish the asymptotic normality of a quadratic form in i.i.d. random variables which has a nonvanishing diagonal. Our theory covers the case of both the finite and the infinite fourth moment, and leads to new results also in the case of a vanishing diagonal.

MSC: 62E20; 60F05

Keywords: Asymptotic normality; Quadratic form

1. Introduction

Asymptotic statistical inference for a linear process

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \tag{1.1}$$

often requires establishing the limit distribution of the quadratic form

$$Q_{n,X} = \sum_{k,t=1}^{n} d_n(k-t) X_k X_t \tag{1.2}$$

with the kernel $d_n(\cdot)$ depending on the sample size n. A convenient tool for deriving this asymptotic distribution is to approximate $Q_{n,X}$ by the quadratic form

$$Q_{n,Z} = \sum_{k,t=1}^{n} e_n(k-t) Z_k Z_t, \tag{1.3}$$

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in i.i.d. variables Z_t. In such an approximation, the relation between the kernels $d_n(\cdot)$ and $e_n(\cdot)$ is given by

$$d_n(t) = \int_{-\pi}^{\pi} \eta_n(\lambda)\,\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}\lambda, \quad e_n(t) = 2\pi \int_{-\pi}^{\pi} \eta_n(\lambda) f(\lambda)\,\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}\lambda, \tag{1.4}$$

where $\eta_n(\cdot)$, $n = 1, 2, 3, \ldots$, is a sequence of even real functions and $f(\cdot)$ is the spectral density of the process $\{X_t\}$. Phillips and Solo (1992), Mikosch et al. (1995), Klüppelberg and Mikosch (1996), Kokoszka and Taqqu (1996), Horváth and Shao (1999) and Fay et al. (2002), among others, used this technique with kernels $d(\cdot)$ and $e(\cdot)$ which do not depend on n. Motivated by applications involving parametric estimation, model fitting and semiparametric inference, Bhansali et al. (2007) developed a general approach to approximating $Q_{n,X}$ by $Q_{n,Z}$ for η_n depending on n which leads to an asymptotic theory for $Q_{n,X}$ under weak conditions. It requires a central limit theorem (CLT) for the form $Q_{n,Z}$, which is established in this paper together with some extensions and corrolaries.

The form $Q_{n,Z}$ is seen to have a nonvanishing diagonal, yet to our knowledge no general results for quadratic forms with nonvanishing diagonal are available at present. For this reason, direct asymptotic inference based on the form $Q_{n,X}$, which is a weighted periodogram, involves complex and tedious calculations. Moreover, the existing results cover the cases of the innovations Z_t either with finite fourth moment or with infinite second moment, see the references following (1.4).

The objective of this paper is to develop a useful limit theory for quadratic forms in i.i.d. random variables which allows: (1) non-zero diagonal, (2) infinite fourth and finite second moment, (3) the coefficients which depend on the dimension of the kernel matrix.

The main results with assumptions and some additional discussion are stated in Section 2. The proofs are developed in Section 3.

2. Main results

In this paper we consider the general quadratic form

$$T_n = \sum_{t,k=1}^{n} a_{n;tk} Z_t Z_k,$$

where $\{Z_k\}$ are i.i.d. variables, $EZ_k = 0, EZ_k^2 = 1$ and $a_{n;tk}$ are entries of a real symmetric matrix $A_n = (a_{n;tk})_{t,k=1,\ldots,n}$. Define $\tilde{A}_n = (\tilde{a}_{n;tk})_{t,k=1,\ldots,n}$ by setting $\tilde{a}_{n;tk} = a_{n;tk}$, if $1 \leqslant k < t \leqslant n$; $\tilde{a}_{n;tk} = 0$, if $1 \leqslant t \leqslant k \leqslant n$. The matrix $\tilde{A}_n$ is a triangular projection of the matrix A_n with diagonal entries equal to zero. We shall denote by $\tilde{A}_n'$ its transposed matrix. Denote by $||A_n|| = (\sum_{t,k=1}^{n} a_{n;tk}^2)^{1/2}$ the Euclidean norm and by $||A_n||_{\mathrm{sp}} = \max_{||x||=1} ||A_n x||$ the spectral norm of the matrix A_n.

The CLT for quadratic forms T_n is well investigated in case when A_n is a symmetric matrix with vanishing diagonal elements: $a_{n;tt} = 0$ for all t. Then the condition $||A_n||_{\mathrm{sp}} = \mathrm{o}(||A_n||)$ is sufficient for the CLT to hold, see Rotar (1973), Jong (1987), Guttorp and Lockhart (1988) and Mikosch (1991). The main objective of this paper is to extend these results to the case of non-zero diagonal elements.

In Theorem 2.1(i), assuming finite fourth moment, we thus show that the CLT extends to the case when A_n has non-zero diagonal elements. We also replace the sufficient condition $||A_n||_{\mathrm{sp}} = \mathrm{o}(||A_n||)$ by a weaker new bound involving the Euclidean norm of the product $\tilde{A}_n'\tilde{A}_n$ of triangular projection $\tilde{A}_n$. This new bound is of central importance in the proofs. Its relation to the bound $||A_n||_{\mathrm{sp}} = \mathrm{o}(||A_n||)$ is stated as Lemma 2.1, following the statement of Theorem 2.1. Part (ii) is a more traditional extension of the existing results, it follows from Lemma 2.1 and part (i). Part (iii) of Theorem 2.1 focuses on the case of Z_t with possibly infinite fourth moment. We show that if a non-zero diagonal vanishes at the rate $\sum_{t=1}^{n} a_{n;tt}^2 = \mathrm{o}(||A_n||^2)$, then the CLT holds under $2+\delta$ finite moments, whereas in the case of the zero diagonal only two finite moments are needed.

The proofs are based on the martingale CLT, but require delicate work to determine how many moments of Z_t are needed for the CLT to hold. The proofs of Theorems 2.1 and 2.2 and of Lemma 2.1 are given in Section 3.

The following quantity plays an important role in our theory:

$$\Delta_n = \frac{||\tilde{A}_n'\tilde{A}_n|| + \max_{t=1,\dots,n}\sum_{s=1}^{n} a_{ts}^2}{||A_n||^2}. \tag{2.1}$$

Theorem 2.1. (i) *If* $EZ_t^4 < \infty$ *and*

$$\Delta_n \to 0, \tag{2.2}$$

then

$$(Var(T_n))^{-1/2}(T_n - ET_n) \xrightarrow{\mathrm{d}} N(0,1). \tag{2.3}$$

(ii) *If* $EZ_t^4 < \infty$ *and*

$$\frac{||A_n||_{\mathrm{sp}}}{||A_n||} \to 0, \tag{2.4}$$

then convergence (2.3) *holds.*

(iii) *Assume that* (2.2) *or* (2.4) *is satisfied and either*

$$EZ_t^2 < \infty \quad \textit{and} \quad a_{n;tt}^2 = 0, \quad t = 1,\dots,n, \tag{2.5}$$

or

$$EZ_t^{2+\delta} < \infty \quad (\textit{for some } \delta > 0) \quad \textit{and} \quad \sum_{t=1}^{n} a_{n;tt}^2 = \mathrm{o}(||A_n||^2) \tag{2.6}$$

holds. Then

$$\frac{1}{\sqrt{2}||A_n||}(T_n - ET_n) \xrightarrow{\mathrm{d}} N(0,1). \tag{2.7}$$

The following lemma shows that assumption (2.4) is stronger than assumption (2.2).

Lemma 2.1. *For any symmetric matrix* A_n,

$$||\tilde{A}_n'\tilde{A}_n|| \leqslant C||A_n||_{\mathrm{sp}}||A_n|| \tag{2.8}$$

and

$$\Delta_n \leqslant C\frac{||A_n||_{\mathrm{sp}}}{||A_n||}, \tag{2.9}$$

where C does not depend on n.

Next we consider the case when A_n is a Toeplitz matrix with entries

$$a_{n;tk} = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(t-k)x} g_n(x)\,\mathrm{d}x, \quad t,k = 1,\dots,n, \tag{2.10}$$

where $g_n(x)$, $|x| \leqslant \pi$ is an even real function. Then Δ_n can be evaluated in terms of the function $g_n(\cdot)$. Assumptions on the function $g_n(\cdot)$ are tailored to facilitate asymptotic inference for processes whose spectral density follows a power law at the origin.

Theorem 2.2. *Let* A_n *be a Toeplitz matrix with entries* $a_{n;tk}$ *given by* (2.10). *Assume that there exist* $0 \leqslant \alpha < 1$ *and a sequence of constants* $k_n > 0$ *such that uniformly in* $|\lambda| \leqslant \pi$,

$$|g_n(\lambda)| \leqslant k_n|\lambda|^{-\alpha}, \quad n \geqslant 1. \tag{2.11}$$

(i) *Then*

$$||A_n||_{\mathrm{sp}} \leqslant Ck_n n^{\alpha}, \quad n \geqslant 1. \tag{2.12}$$

(ii) *Moreover, if*

$$\frac{k_n n^{\alpha}}{||A_n||} \to 0 \tag{2.13}$$

and $EZ_t^4 < \infty$ *then the quadratic form* T_n *satisfies the CLT* (2.3), *whereas if* Z_t *and* $a_{n;tt}$ *satisfy either* (2.5) *or* (2.6) *then the CLT* (2.7) *holds.*

Theorem 2.2 implies that if $|g_n(x)| \leqslant C$, for $|x| \leqslant |\pi|$ and $n \geqslant 1$, and $Var(T_n) \to \infty$, then (2.13) is satisfied in view of (3.2) below, and the CLT holds.

For the CLT to be valid, it is natural to assume that (2.11) holds with $\alpha \in [0, 1/2)$. For example, if $g_n(x) \equiv g(x) = |x|^{-\alpha} h(x)$ where $0 < h(x) < \infty$ is a continuous bounded function, then $||A_n|| \sim Cn^{\max(1/2,\alpha)}$ if $0 \leqslant \alpha < 1$ and $\alpha \neq 1/2$. Then by (2.12), $\Delta_n \leqslant C||A_n||_{\mathrm{sp}}/||A_n|| = o(1)$ if $0 \leqslant \alpha < 1/2$ and therefore CLT holds. However, if $1/2 < \alpha < 1$ then (2.12) gives the bound $\Delta_n \leqslant C$ which does not imply the CLT. It is well known that in this case the CLT does not hold and T_n satisfies a non-CLT, see Giraitis et al. (1988).

3. Proofs

Proof of Theorem 2.1. For simplicity we shall write a_{ts} instead of $a_{n;ts}$.

The variance of T_n is

$$Var(T_n) = 2 \sum_{t,k=1:t \neq k}^{n} a_{n;tk}^2 + Var(Z_0^2) \sum_{t=1}^{n} a_{n,tt}^2(0) \tag{3.1}$$

and so

$$Var(T_n) \asymp ||A_n||^2. \tag{3.2}$$

Recall that $a_n \asymp b_n$ means that there exist constants c_1 and c_2 such that $c_1 a_n \leqslant b_n \leqslant c_2 a_n$ as $n \to \infty$.

Proof of (i) *and* (ii). By Lemma 2.1, condition (2.4) implies (2.2). Therefore it suffices to show that CLT (2.3) holds under assumption (2.2).

Write

$$T_n - ET_n = \sum_{t=1}^{n} v_t,$$

where

$$v_t = 2Z_t \sum_{s=1}^{t-1} a_{ts} Z_s + a_{tt}(Z_t^2 - EZ_t^2), \quad t \geqslant 2, \quad v_1 = a_{11}(Z_1^2 - EZ_1^2). \tag{3.3}$$

Set $B_n = Var(T_n)$. Since $v_t = v_t(n)$ is a zero mean martingale difference array with respect to the sigma algebra $\mathscr{F}_{t-1}$ generated by the variables $\{Z_s, 1 \leqslant s \leqslant t-1\}$, to show convergence (2.3), it suffices to prove that, see Corollary 3.1 of Hall and Heyde (1980),

$$S_n := B_n^{-1} \sum_{t=1}^{n} \mathrm{E}[v_t^2 | \mathscr{F}_{t-1}] \xrightarrow{\mathrm{d}} 1 \tag{3.4}$$

and

$$q_n(\delta) := B_n^{-1} \sum_{t=1}^{n} \mathrm{E}\left[v_t^2 1_{\{|v_t| \geqslant \delta B_n^{1/2}\}}\right] \to 0 \quad \text{for all } \delta > 0. \tag{3.5}$$

First we show (3.4). Put

$$\tau_n(t) = 2 \sum_{s=1}^{t-1} a_{ts} Z_s, \quad C_1 = E[Z_t(Z_t^2 - EZ_t^2)], \quad C_2 = Var(Z_t^2) \equiv E(Z_t^2 - EZ_t^2)^2.$$

Then

$$S_n := B_n^{-1} \sum_{t=1}^{n} \mathrm{E}[v_t^2 | \mathscr{F}_{t-1}] = B_n^{-1} \sum_{t=1}^{n} \big(4\tau_n^2(t) + 4C_1 a_{tt}\tau_n(t) + C_2 a_{tt}^2\big).$$

We have that

$$ES_n = B_n^{-1}\left(\sum_{t=1}^{n}\left[4\sum_{s=1}^{t-1} a_{ts}^2 + C_2 a_{tt}^2\right]\right) = 1.$$

Therefore

$$\begin{aligned}|S_n - 1| &\leqslant CB_n^{-1}\left[\left|\sum_{t=1}^{n}(\tau_n^2(t) - E\tau_n^2(t))\right| + \left|\sum_{t=1}^{n} a_{tt}\tau_n(t)\right|\right] \\ &\leqslant CB_n^{-1}\big[|S_{n,1}| + |S_{n,2}| + |S_{n,3}|\big],\end{aligned}$$

where

$$S_{n,1} = \sum_{t=1}^{n} \sum_{1 \leqslant s_1 < s_2 \leqslant t-1} a_{ts_1} a_{ts_2} Z_{s_1} Z_{s_2}, \quad S_{n,2} = \sum_{t=1}^{n} \sum_{1 \leqslant s \leqslant t-1} a_{ts}^2 (Z_s^2 - EZ_s^2)$$

and $S_{n,3} = \sum_{t=1}^{n} a_{tt}\tau_n(t)$. Then

$$ES_{n,1}^2 \leqslant C \sum_{t_1,t_2=1}^{n} \sum_{1 \leqslant s_1 < s_2 < \min(t_1,t_2)} a_{t_1 s_1} a_{t_2 s_1} a_{t_1 s_2} a_{t_2 s_2}$$

and

$$ES_{n,2}^2 \leqslant C \sum_{t_1,t_2=1}^{n} \sum_{1 \leqslant s < \min(t_1,t_2)} a_{t_1 s}^2 a_{t_2 s}^2,$$

which shows that

$$ES_{n,1}^2 + ES_{n,2}^2 \leqslant C \sum_{t_1,t_2=1}^{n} \sum_{1 \leqslant s_1,s_2 < \min(t_1,t_2)} a_{t_1 s_1} a_{t_2 s_1} a_{t_1 s_2} a_{t_2 s_2} = C||\tilde{A}_n' \tilde{A}_n||^2.$$

On the other hand,

$$\begin{aligned}ES_{n,3}^2 &\leqslant C \sum_{t_1,t_2=1}^{n} a_{t_1 t_1} a_{t_2 t_2} \sum_{1 \leqslant s < \min(t_1,t_2)} a_{t_1 s} a_{t_2 s} \\ &\leqslant C\left(\sum_{t=1}^{n} a_{tt}^2\right)\left(\sum_{t_1,t_2=1}^{n}\left|\sum_{1 \leqslant s < \min(t_1,t_2)} a_{t_1 s} a_{t_2 s}\right|^2\right)^{1/2} \\ &\leqslant C\left(\sum_{t=1}^{n} a_{tt}^2\right)||\tilde{A}_n' \tilde{A}_n|| \leqslant C||A_n||^2 ||\tilde{A}_n' \tilde{A}_n||.\end{aligned}$$

Thus

$$\begin{aligned}E(S_n - 1)^2 &\leqslant CB_n^{-2}(ES_{n,1}^2 + ES_{n,2}^2 + ES_{n,3}^2) \\ &\leqslant CB_n^{-2}(||\tilde{A}' \tilde{A}||^2 + ||A_n||^2 ||\tilde{A}_n' \tilde{A}_n||) \leqslant C(\Delta_n^2 + \Delta_n) \to 0\end{aligned}$$

by (2.2) and (3.2), to prove (3.4).

We first prove (3.5) assuming $EZ_j^8 < \infty$. Then

$$q_n(\delta) \leqslant B_n^{-2}\delta^{-2} \sum_{t=1}^{n} Ev_t^4.$$

From (3.3) it follows that

$$v_t^4 \leqslant C\left(Z_t^4 \left| \sum_{s=1}^{t-1} 2a_{ts} Z_s \right|^4 + a_{tt}^4 (Z_t^2 - EZ_t^2)^4 \right),$$

where

$$\begin{aligned}\left| \sum_{s=1}^{t-1} a_{ts} Z_s \right|^4 &\leqslant \left| \sum_{1 \leqslant s_1, s_2 \leqslant t-1} a_{ts_1} a_{ts_2} Z_{s_1} Z_{s_2} \right|^2 \\ &\leqslant C\left(\left| \sum_{1 \leqslant s_1 < s_2 \leqslant t-1} a_{ts_1} a_{ts_2} Z_{s_1} Z_{s_2} \right|^2 + \left| \sum_{1 \leqslant s \leqslant t-1} a_{ts}^2 Z_s^2 \right|^2 \right).\end{aligned}$$

Hence

$$\begin{aligned} q_n(\delta) &\leqslant CB_n^{-2} \sum_{t=1}^n E\left[\left| \sum_{1 \leqslant s_1 < s_2 \leqslant t-1} a_{ts_1} a_{ts_2} Z_{s_1} Z_{s_2} \right|^2 + \left| \sum_{1 \leqslant s \leqslant t-1} a_{ts}^2 Z_s^2 \right|^2 + a_{tt}^4 \right] \\ &\leqslant CB_n^{-2} \sum_{t=1}^n \left[\sum_{1 \leqslant s_1, s_2 \leqslant t-1} a_{ts_1}^2 a_{ts_2}^2 + a_{tt}^4 \right] \leqslant CB_n^{-2} ||A||^2 \max_{t=1,\ldots,n} \sum_{s=1}^n a_{ts}^2 \\ &\leqslant C\Delta_n \to 0 \end{aligned}$$

by (2.2) and (3.2).

Now we show that (3.5) is valid under the assumption $EZ_j^4 < \infty$. Let $K > 0$. Set

$$Z_t^- = Z_t 1_{\{|Z_t| \leqslant K\}} - E[Z_t 1_{\{|Z_t| \leqslant K\}}], \quad Z_t^+ = Z_t 1_{\{|Z_t| > K\}} - E[Z_t 1_{\{|Z_t| > K\}}]. \tag{3.6}$$

Then (Z_t^-) and (Z_t^+) are sequences of i.i.d. variables with zero mean, and $Z_t = Z_t^- + Z_t^-$. Rewrite v_t given by (3.3) as

$$v_t = v_t^- + v_t^+,$$

where

$$v_t^- = 2\sum_{s=1}^{t-1} a_{ts} Z_t^- Z_s^- + a_{tt}((Z_t^-)^2 - E(Z_t^-)^2), \quad v_t^+ = v_t - v_t^-.$$

Note that if $|v_t^-| \leqslant |v_t^+|/2$ then $v_t^2 \leqslant (|v_t^-| + |v_t^+|)^2 \leqslant (3|v_t^+|/2)^2$, whereas if $|v_t^-| > |v_t^+|/2$ then $v_t^2 \leqslant (|v_t^-| + |v_t^+|)^2 \leqslant (3|v_t^-|)^2$, and in addition, $1_{\{|v_t| \geqslant \delta B_n^{1/2}\}} \leqslant 1_{\{|3v_t^-| \geqslant \delta B_n^{1/2}\}}$. Hence

$$\begin{aligned} q_n(\delta) &= B_n^{-1} \sum_{t=1}^n \mathrm{E}\left[v_t^2 1_{\{|v_t| \geqslant \delta B_n^{1/2}\}} \right] \\ &\leqslant B_n^{-1} \sum_{t=1}^n \mathrm{E}\left[(3v_t^-)^2 1_{\{3|v_t^-| \geqslant \delta B_n^{1/2}\}} \right] + B_n^{-1} \sum_{t=1}^n \mathrm{E}[(3v_t^+/2)^2] \\ &=: q_n^-(\delta) + q_n^+(\delta). \end{aligned}$$

Since variables Z_t^- have all finite moments, then as we have seen above, $q_n^-(\delta) \to 0$, as $n \to \infty$, for any $K > 0$. To complete the proof of (3.5) it suffices to show that

$$\sup_{n \geqslant 1} q_n^+(\delta) \to 0 \quad \text{as } K \to \infty.$$

Since

$$Z_t Z_s - Z_t^- Z_s^- = Z_t^+ Z_s^+ + Z_t^- Z_s^+ + Z_t^+ Z_s^-, \quad (Z_t)^2 - (Z_t^-)^2 = 2Z_t^- Z_t^+ + (Z_t^+)^2,$$

then it is easy to see that

$$q_n^+(\delta) \leqslant CB_n^{-1}E\left(\left|\sum_{t=1}^{n}\sum_{s=1}^{t-1}a_{ts}Z_t^+Z_s^+\right|^2+\left|\sum_{t=1}^{n}\sum_{s=1}^{t-1}a_{ts}Z_t^-Z_s^+\right|^2\right.$$
$$\left.+\left|\sum_{t=1}^{n}\sum_{s=1}^{t-1}a_{ts}Z_t^+Z_s^-\right|^2+\left|\sum_{t=1}^{n}a_{tt}\left((Z_t)^2-(Z_t^-)^2-E[(Z_t)^2-(Z_t^-)^2]\right)\right|^2\right)$$
$$\leqslant CB_n^{-1}||A_n||^2\delta_K \leqslant C\delta_K \to 0$$

since $||A_n||^2 \leqslant CB_n$ by (3.2), and

$$\delta_K := E|Z_t^+|^2 + (E|Z_t^+|^4)^{1/2} \to 0 \quad \text{as } K \to \infty.$$

Proof of (iii). Suppose that $a_{n;tt}^2 = 0, \quad t = 1, \dots, n$, and $EZ_t^2 < \infty$. Then using truncation (3.6), by the same argument as above it can be shown that (3.4) and (3.5) are satisfied with $B_n = 2||A_n||^2$, which implies (2.7).

If $\sum_{t=1}^n a_{n;tt}^2 = \mathrm{o}(||A||^2)$, then we can write

$$T_n - ET_n = \sum_{t,k=1}^{n} a_{n;tk}Z_tZ_k = \sum_{t,k=1:t\neq k}^{n} a_{n;tk}Z_tZ_k + \sum_{t=1}^{n} a_{n;tt}(Z_t^2 - EZ_t^2) =: \bar{T}_n + T_n^*.$$

Denote $\bar{A}_n = (\bar{a}_{n;ts})_{t,s=1,\dots,n}$ where $\bar{a}_{n;ts} = a_{n;ts}$ if $t \neq s$, $\bar{a}_{n;tt} = 0$. By assumption (2.6), $||\bar{A}_n|| = ||A_n||(1 + \mathrm{o}(1))$, and we have shown above that

$$(\sqrt{2}||A_n||)^{-1}(\bar{T}_n - E\bar{T}_n) \xrightarrow{\mathrm{d}} N(0,1).$$

Then (2.7) follows if we show that

$$||A_n||^{-1}T_n^* \xrightarrow{\mathrm{d}} 0. \tag{3.7}$$

To estimate T_n^* we use the inequality

$$E\left|\sum_{t=1}^{n} a_tY_t\right|^p \leqslant E|Y_1|^p\left|\sum_{t=1}^{n} a_t^2\right|^{p/2}, \quad p > 1,$$

which is valid for any real numbers a_t and i.i.d. random variables (Y_t), such that $EY_t = 0$ and $E|Y_1|^p < \infty$, see Lemma 1.3 of Mikosch (1991). Setting $Y_t = Z_t^2 - E[Z_t^2]$ and noting that by assumption (2.6), $E|Y_1|^p < \infty$ with $p = 1 + \delta/2$, we obtain that

$$E|\,||A_n||^{-1}T_n^*|^p = E\left|||A_n||^{-1}\sum_{t=1}^{n} a_{n,tt}(Z_t^2 - E[Z_t^2])\right|^p \leqslant C\left|||A_n||^{-2}\sum_{t=1}^{n} a_{n:tt}^2\right|^{(1+\delta/2)/2} \to 0,$$

in view of (2.6), to prove (3.7). $\square$

Proof of Lemma 2.1. We first list some properties of matrices which are used in the proof. Let B be a real $n \times n$ matrix. Then $||B||^2 = \mathrm{Tr}[B'B]$, where Tr denotes the trace, the sum of the diagonal elements. In particular, if B is symmetric, then $||B||^2 = \mathrm{Tr}[B^2]$. The matrix $B'B$ is symmetric and positive definite, and has nonnegative eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. For every $p \geqslant 1$,

$$||B||_p = \left[\sum_{i=1}^{n} \lambda_i^{p/2}\right]^{1/p} \tag{3.8}$$

defines the *Schatten l_p norm*. The matrix $B'B$ has a root $|B|$, which is a symmetric matrix satisfying $|B|^2 = B'B$. Observe that $||B||_4^4 = ||\,|B|\,||_4^4 = \sum_{i=1}^n \lambda_i^2$. Since $B'B$ is symmetric and nonnegative definite, there is an orthonormal matrix U such that

$$U'B'BU = \mathrm{diag}(\lambda_1, \dots, \lambda_n).$$

Therefore

$$\sum_{j=1}^{n} \lambda_j^2 = \mathrm{Tr}[(U'B'BU)^2] = \mathrm{Tr}[U'(B'B)^2U] = \mathrm{Tr}[(B'B)^2]$$

and so we obtain the identities

$$||B||_4^4 = \mathrm{Tr}[|B|^4] = \mathrm{Tr}[(B'B)^2] = ||B'B||_2^2. \tag{3.9}$$

Since $||B||_{\mathrm{sp}} = \lambda_1^{1/2}$ and $||B||_2^2 = \sum_{i=1}^{n} \lambda_i$, we also have

$$||B'B||_2^2 = \sum_{i=1}^{n} \lambda_i^2 \leqslant \lambda_1 \sum_{i=1}^{n} \lambda_i = ||B||_{\mathrm{sp}}^2 ||B||_2^2. \tag{3.10}$$

Finally, it is known that

$$||\tilde{B}||_p \leqslant C_p ||B||_p, \tag{3.11}$$

where C_p does not depend on B, see e.g. Macaev (1961) and Nikolski (2002, p. 278).

Using the above properties observe that

$$||\tilde{A}_n'\tilde{A}_n||^2 = |||\tilde{A}_n|^2||^2 = \mathrm{Tr}[|\tilde{A}_n|^4] = ||\,|\tilde{A}_n|\,||_4^4$$

and

$$||\,|\tilde{A}_n|\,||_4^4 \leqslant C_4^4 ||A_n||_4^4 = C_4^4 ||A_n'A_n||_2^2 \leqslant C_4^4 ||A_n||_{\mathrm{sp}}^2 ||A_n||_2^2,$$

which implies (2.8). Relation (2.9) follows from (2.8) and the well-known estimate

$$\max_{t=1,\ldots,n} \sum_{s=1}^{n} a_{ts}^2 \leqslant ||A_n||_{\mathrm{sp}}^2. \quad \square$$

Proof of Theorem 2.2. (i) Write

$$g_n(y) = g_n(y)1(|y| \leqslant n^{-1}) + g_n(y)1(|y| > n^{-1}) =: g_n^-(y) + g_n^+(y).$$

Set

$$a_{tk} = \int_{-\pi}^{\pi} g_n(y)\,\mathrm{e}^{\mathrm{i}(t-k)y}\,\mathrm{d}y = \int_{-\pi}^{\pi} g_n^-(y)\,\mathrm{e}^{\mathrm{i}(t-k)y}\,\mathrm{d}y + \int_{-\pi}^{\pi} g_n^+(y)\,\mathrm{e}^{\mathrm{i}(t-k)y}\,\mathrm{d}y =: a_{tk}^- = \ a_{tk}^+,$$

and define $A_n^{\pm} = (a_{ts}^{\pm})_{t,s=1,\ldots,n}$. Then

$$\begin{aligned} ||A_n||_{\mathrm{sp}} &= ||A_n^- + A_n^+||_{\mathrm{sp}} \leqslant ||A_n^-||_{\mathrm{sp}} + ||A_n^+||_{\mathrm{sp}} \\ &= \sup_{||x||=1} \left(\sum_{t=1}^{n} \left| \sum_{s=1}^{n} a_{ts}^- x_s \right|^2 \right)^{1/2} + \sup_{||x||=1} \left(\sum_{t=1}^{n} \left| \sum_{s=1}^{n} a_{ts}^+ x_s \right|^2 \right)^{1/2}. \end{aligned}$$

By Parseval's equality, for any $||x|| = 1$,

$$\begin{aligned} \sum_{t=1}^{n} \left| \sum_{s=1}^{n} a_{ts}^+ x_s \right|^2 &\leqslant C \sum_{t=-\infty}^{\infty} \left| \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}ty} \left(g_n^+(y) \sum_{s=1}^{n} \mathrm{e}^{-\mathrm{i}sy} x_s \right) \mathrm{d}y \right|^2 \\ &\leqslant C \int_{-\pi}^{\pi} |g_n^+(y)|^2 \left| \sum_{s=1}^{n} \mathrm{e}^{\mathrm{i}sy} x_s \right|^2 \mathrm{d}y \leqslant Ck_n^2 \int_{-\pi}^{\pi} |y|^{-2\alpha} 1(|y| > n^{-1}) \left| \sum_{s=1}^{n} \mathrm{e}^{\mathrm{i}sy} x_s \right|^2 \mathrm{d}y \\ &\leqslant Ck_n^2 n^{2\alpha} \int_{-\pi}^{\pi} \left| \sum_{s=1}^{n} \mathrm{e}^{\mathrm{i}sy} x_s \right|^2 \mathrm{d}y \leqslant Ck_n^2 n^{2\alpha} ||x||^2 = Ck_n^2 n^{2\alpha}. \end{aligned}$$

On the other hand,

$$\sum_{t=1}^{n}\left|\sum_{s=1}^{n} a_{ts}^{-}x_s\right|^2 \leqslant \sum_{t=1}^{n}\left|\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}ty}\left(g_n^{-}(y)\sum_{s=1}^{n}\mathrm{e}^{-\mathrm{i}sy}x_s\right)\mathrm{d}y\right|^2$$
$$\leqslant \sum_{t=1}^{n}\left(\int_{|y|\leqslant n^{-1}} k_n|y|^{-\alpha}\sum_{s=1}^{n}|x_s|\,\mathrm{d}y\right)^2 \leqslant Ck_n^2 n^{2\alpha},$$

since $\sum_{s=1}^{n}|x_s|\leqslant n^{1/2}||x||\leqslant n^{1/2}$, to complete the proof.

Part (ii) follows from Theorem 2.1, using (2.9) and (2.12). □

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